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Blumenthal's Theorem for Laurent Orthogonal Polynomials[1]

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We investigate polynomials satisfying a three-term recurrence relation of the form $B_n(x) = (x - \beta_n)B_{n-1}(x) - \alpha_n x B_{n-2}(x)$, with positive recurrence coefficients α_{n+1}, β_n $(n = 1, 2, \ldots)$. We show that the zeros are eigenvalues of a structured Hessenberg matrix and give the left and right eigenvectors of this matrix, from which we deduce Laurent orthogonality and the Gaussian quadrature formula. We analyse in more detail the case where $\alpha_n \to \alpha$ and $\beta_n \to \beta$ and show that the zeros of B_n are dense on an interval and that the support of the Laurent orthogonality measure is equal to this interval and a set which is at most denumerable with accumulation points (if any) at the endpoints of the interval. This result is the Laurent version of Blumenthal's theorem for orthogonal polynomials.

1. INTRODUCTION

We will investigate families of polynomials satisfying a three-term recurrence relation of the form

$$B_n(x) = (x - \beta_n)B_{n-1}(x) - \alpha_n x B_{n-2}(x), \qquad n \geqslant 1, \tag{1.1}$$

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with $B_0 = 1$ and $B_{-1} = 0$. It was shown in [6] that if $\alpha_n > 0$ for every $n \geqslant 2$ and $\beta_n > 0$ for every $n \geqslant 1$, then the polynomials B_n are denominator polynomials for a positive T-fraction (M-fraction) and as such they satisfy a certain orthogonality property:

$$\int x^{-n+k} B_n(x)\, d\mu(x) = 0, \qquad k = 0, 1, \ldots, n-1, \tag{1.2}$$

where μ is some positive measure on the positive real line for which all the strong moments

$$\int x^k\, d\mu(x), \qquad k \in \mathbb{Z}$$

exist. Furthermore, the zeros of B_n are real and simple, they are all on the positive real line and they interlace with the zeros of B_{n-1}. These properties easily follow from the recurrence relation (1.1), which shows that we are dealing with a Sturm sequence. Indeed, we have that sign $B_n(0) = (-1)^n$ and at the zeros $x_{j,n-1}$ of B_{n-1} we have $B_n(x_{j,n-1}) = -\alpha_n x_{j,n-1} B_{n-2}(x_{j,n-1})$, so that $B_n(x_{j,n-1})$ and $B_{n-2}(x_{j,n-1})$ have opposite sign if $x_{j,n-1}$ is positive. The properties of the zeros follow from this by induction. These properties also follow from (1.2) which implies that B_n is the orthogonal polynomial of degree n for the *varying* measure $d\mu_n(x) = x^{-n}\, d\mu(x)$, and therefore the zeros of B_n have the usual properties of orthogonal polynomials. Our objective is to obtain some information about this measure μ and its support when some information about the recurrence coefficients $\{\alpha_{n+1}, \beta_n,\ n = 1, 2, 3, \ldots\}$ is given.

Laurent orthogonal polynomials are of interest for a number of reasons. First of all they correspond to three-term recurrence relations of the form (1.1) which are of a different nature than the usual three-term recurrence relations of orthogonal polynomials. The Laurent orthogonality (1.2) and recurrence (1.1) then give rise to inverse and direct spectral problems. Secondly, Laurent orthogonal polynomials are a limiting case of orthogonal rational functions (see [4]) when all the poles of the rational functions are at the origin. Usual orthogonal polynomials are another limiting case that occurs when all the poles tend to infinity. These limiting cases require a separate treatment and as such stand out from the general case. Laurent orthogonal polynomials are related to the strong Stieltjes moment problem [6] and a special kind of continued fractions, known as positive T-fractions (Thron fractions). Finally, it turns out that Laurent orthogonal polynomials play an important role in the analysis of the relativistic Toda lattice, which was introduced by Ruijsenaars [10]. This role is similar to the role of the usual orthogonal polynomials in the analysis of the Toda lattice. See, for instance [17]. The direct and inverse spectral problems for Laurent orthogonal polynomials are very natural in this application.

In this paper, we will investigate the case when

$$\lim_{n\to\infty} \alpha_n = \alpha > 0, \qquad \lim_{n\to\infty} \beta_n = \beta > 0. \tag{1.3}$$

We will obtain some information about the Laurent orthogonality measure μ using this information on the recurrence coefficients. The situation is well known in the case of orthogonal polynomials p_n $(n = 0, 1, 2, \ldots)$ satisfying a three-term recurrence relation

$$x p_n(x) = a_{n+1} p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x), \qquad n \geqslant 0, \tag{1.4}$$

with $p_0 = 1$ and $p_{-1} = 0$. When the recurrence coefficients for these polynomials converge,

$$\lim_{n\to\infty} a_n = a/2 \geqslant 0, \qquad \lim_{n\to\infty} b_n = 0, \tag{1.5}$$

then the corresponding family of orthogonal polynomials, or equivalently their orthogonality measure v for which

$$\int p_n(x) p_m(x)\, dv(x) = \delta_{m,n}, \qquad m, n \geqslant 0,$$

belongs to the class $M(a, b)$, which has been studied intensively during the past two decades [9, 16]. A century ago, Blumenthal [3] proved that (1.5) implies that the zeros of the orthogonal polynomials p_n $(n = 1, 2, \ldots)$ are dense in the interval $[b - a, b + a]$. Later it became clear that Blumenthal's result in fact gives relevant information about the support of the orthogonality measure v (see, e.g., [8]): indeed (1.5) implies that the interval $[b - a, b + a]$ is a subset of the support of the measure v and that $\mathrm{supp}(v) \setminus [b - a, b + a]$ is at most denumerable with accumulation points (if any) at the endpoints $b \pm a$. In this paper, we will obtain a similar result for Laurent orthogonal polynomials B_n $(n = 1, 2, 3, \ldots)$ satisfying a three-term recurrence relation (1.1) with recurrence coefficients for which (1.3) holds. The situation is somewhat different from the case of orthogonal polynomials in the sense that the recurrence relation (1.1) does not give rise to a self-adjoint operator, whereas the recurrence relation (1.4) gives a symmetric tridiagonal Jacobi matrix which can be extended to a self-adjoint operator acting on ℓ_2. The case where the recurrence coefficients are asymptotically periodic with period two, i.e., when the subsequences of the recurrence coefficients with even indices and the odd indices are converging, was recently considered in [1], where the ratio asymptotics and the asymptotic zero distribution was obtained.

2. MATRIX APPROACH

Consider the Laurent polynomials

$$\tilde{B}_n(x) = x^{-n} B_n(x),$$

then the recurrence formula (1.1) becomes

$$x\tilde{B}_n(x) = (x - \beta_n)\tilde{B}_{n-1}(x) - \alpha_n \tilde{B}_{n-2}(x),$$

which can be rewritten as

$$x[\tilde{B}_{n-1}(x) - \tilde{B}_n(x)] = \beta_n \tilde{B}_{n-1}(x) + \alpha_n \tilde{B}_{n-2}(x).$$

In matrix notation this becomes

$$\begin{pmatrix} \beta_1 & 0 & 0 & \cdots & 0 \\ \alpha_2 & \beta_2 & 0 & \cdots & 0 \\ 0 & \alpha_3 & \beta_3 & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \alpha_n & \beta_n \end{pmatrix} \begin{pmatrix} \tilde{B}_0(x) \\ \tilde{B}_1(x) \\ \tilde{B}_2(x) \\ \vdots \\ \tilde{B}_{n-1}(x) \end{pmatrix}$$

$$= x \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & 0 & & 1 & -1 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{B}_0(x) \\ \tilde{B}_1(x) \\ \tilde{B}_2(x) \\ \vdots \\ \tilde{B}_{n-1}(x) \end{pmatrix} - x\tilde{B}_n(x) \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

If we denote the $n \times n$ matrices F and G by

$$F = \begin{pmatrix} \beta_1 & 0 & 0 & \cdots & 0 \\ \alpha_2 & \beta_2 & 0 & \cdots & 0 \\ 0 & \alpha_3 & \beta_3 & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \alpha_n & \beta_n \end{pmatrix},$$

$$G = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & 0 & & 1 & -1 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

and the vector $\tilde{B}(x)$ by

$$\tilde{B}(x) = (\tilde{B}_0, \tilde{B}_1(x), \tilde{B}_2(x), \ldots, \tilde{B}_{n-1}(x))^T,$$

then this matrix relation is written as

$$F\tilde{B}(x) = xG\tilde{B}(x) - x\tilde{B}_n(x)e_n, \tag{2.1}$$

where e_n is the nth unit vector $(0, 0, \ldots, 0, 1)^T$ in $\mathbb{R}^n$. It follows that at each zero $x_{j,n}$ of B_n we have

$$F\tilde{B}(x_{j,n}) = x_{j,n}G\tilde{B}(x_{j,n}),$$

so that each zero $x_{j,n}$ of B_n is an eigenvalue of $G^{-1}F$ with right eigenvector $\tilde{B}(x_{j,n})$. The matrix $G^{-1}F$ is an upper Hessenberg matrix and is of the form

$$G^{-1}F = \begin{pmatrix} \alpha_2+\beta_1 & \alpha_3+\beta_2 & \alpha_4+\beta_3 & \cdots & \alpha_n+\beta_{n-1} & \beta_n \\ \alpha_2 & \alpha_3+\beta_2 & \alpha_4+\beta_3 & \cdots & \alpha_n+\beta_{n-1} & \beta_n \\ 0 & \alpha_3 & \alpha_4+\beta_3 & \cdots & \alpha_n+\beta_{n-1} & \beta_n \\ \vdots & & \ddots & & \vdots & \vdots \\ 0 & 0 & \cdots & \alpha_{n-1} & \alpha_n+\beta_{n-1} & \beta_n \\ 0 & 0 & \cdots & 0 & \alpha_n & \beta_n \end{pmatrix}. \tag{2.2}$$

This matrix approach for Laurent orthogonal polynomials was first observed in [13] and we will use it in Theorem 3.1 to obtain useful bounds on the zeros of B_n.

Next, we consider the polynomials

$$\hat{B}_n(x) = \frac{B_n(x)}{\alpha_{n+1}\alpha_n \cdots \alpha_3\alpha_2},$$

then the recurrence relation for these polynomials becomes

$$\alpha_{n+1}\hat{B}_n(x) = (x - \beta_n)\hat{B}_{n-1}(x) - x\hat{B}_{n-2}(x),$$

which can be rewritten as

$$x[\hat{B}_{n-1}(x) - \hat{B}_{n-2}(x)] = \alpha_{n+1}\hat{B}_n(x) + \beta_n\hat{B}_{n-1}(x).$$

In matrix notation this becomes

$$\hat{B}(x)^T F = x\hat{B}(x)^T G - \alpha_{n+1}\hat{B}_n(x)e_n^T, \tag{2.3}$$

with

$$\hat{B}(x) = (\hat{B}_0, \hat{B}_1(x), \hat{B}_2(x), \ldots, \hat{B}_{n-1}(x))^T,$$

and hence each zero $x_{j,n}$ of B_n is an eigenvalue of FG^{-1} with left eigenvector $\hat{B}(x_{j,n})$.

Combining the results, we see that $\hat{B}(x_{j,n})$ is a left eigenvector of FG^{-1} for the eigenvalue $x_{j,n}$ and $\tilde{B}(x_{j,n})$ is a right eigenvector for this eigenvalue for $G^{-1}F$, or $G\tilde{B}(x_{j,n})$ is a right eigenvector for this eigenvalue for FG^{-1}. It is well known that the left and right eigenvector of two different eigenvalues of a matrix are orthogonal, hence $\hat{B}(x_{i,n})$ and $G\tilde{B}(x_{j,n})$ are orthogonal. In order to find the inner product of the left and right eigenvector for the same eigenvalue, we will derive a formula of Christoffel–Darboux flavour. Multiply (2.1) on the right by the vector $\hat{B}(y)^T$, then this gives

$$F\tilde{B}(x)\hat{B}(y)^T = xG\tilde{B}(x)\hat{B}(y)^T - x\tilde{B}_n(x)e_n\hat{B}(y)^T. \tag{2.4}$$

Similarly, we multiply (2.3) evaluated at y on the left by $\tilde{B}(x)$ to find

$$\tilde{B}(x)\hat{B}(y)^T F = y\tilde{B}(x)\hat{B}(y)^T G - \alpha_{n+1}\hat{B}_n(y)\tilde{B}(x)e_n^T. \tag{2.5}$$

Now take the trace of (2.4) and (2.5) and subtract, then using the fact that $\operatorname{tr} AB = \operatorname{tr} BA$ gives

$$(x-y)\operatorname{tr} G\tilde{B}(x)\hat{B}(y)^T = x\tilde{B}_n(x)\hat{B}_{n-1}(y) - \alpha_{n+1}\hat{B}_n(y)\tilde{B}_{n-1}(x),$$

which gives the desired Christoffel–Darboux formula

$$\begin{aligned}(x-y)\left(\sum_{k=0}^{n-2}[\tilde{B}_k(x) - \tilde{B}_{k+1}(x)]\hat{B}_k(y) + \tilde{B}_{n-1}(x)\hat{B}_{n-1}(y)\right)\\ = x\tilde{B}_n(x)\hat{B}_{n-1}(y) - \alpha_{n+1}\hat{B}_n(y)\tilde{B}_{n-1}(x).\end{aligned} \tag{2.6}$$

The confluent form for $y \to x$ is, by using de l'Hôpital's rule,

$$\sum_{k=0}^{n-2} [\tilde{B}_k(x) - \tilde{B}_{k+1}(x)]\hat{B}_k(x) + \tilde{B}_{n-1}(x)\hat{B}_{n-1}(x) \\ = \alpha_{n+1}\hat{B}_n'(x)\tilde{B}_{n-1}(x) - x\tilde{B}_n(x)\hat{B}_{n-1}'(x). \qquad (2.7)$$

The orthogonality of $\hat{B}(x_{i,n})$ and $G\tilde{B}(x_{j,n})$ for $i \neq j$ now follows immediately by taking $x = x_{j,n}$ and $y = x_{i,n}$ in (2.6). Taking $x = x_{i,n}$ in (2.7) gives

$$\sum_{k=0}^{n-1} [\tilde{B}_k(x_{i,n}) - \tilde{B}_{k+1}(x_{i,n})]\hat{B}_k(x_{i,n}) = \alpha_{n+1}\hat{B}_n'(x_{i,n})\tilde{B}_{n-1}(x_{i,n}),$$

which is positive since at $x_{i,n}$ the polynomials B_n' and B_{n-1} have the same sign due to the interlacing property we mentioned in the Introduction (see also [6] or Section 3). Hence if we normalize the eigenvectors by putting

$$\tilde{b}(x_{i,n}) = \tilde{B}(x_{i,n})/\tilde{B}_{n-1}(x_{i,n}), \qquad \hat{b}(x_{i,n}) = \hat{B}(x_{i,n})/(\alpha_{n+1}\hat{B}_n'(x_{i,n})),$$

then the left eigenvectors $\hat{b}(x_{i,n})$ and the right eigenvectors $G\tilde{b}(x_{j,n})$ are orthogonal if $i \neq j$ and for $i = j$ their inner product is 1. Hence if we put

$$P = \begin{pmatrix} \hat{b}(x_{1,n})^T \\ \hat{b}(x_{2,n})^T \\ \vdots \\ \hat{b}(x_{n,n})^T \end{pmatrix},$$

then

$$P^{-1} = G(\tilde{b}(x_{1,n}) \quad \tilde{b}(x_{2,n}) \quad \cdots \quad \tilde{b}(x_{n,n}))$$

and

$$PFG^{-1}P^{-1} = D,$$

where D is the diagonal matrix containing the eigenvalues $x_{1,n}, x_{2,n}, \ldots, x_{n,n}$.

Observe that as an extra result we have from $P^{-1}P = I$

$$\sum_{j=1}^{n} [\tilde{B}_k(x_{j,n}) - \tilde{B}_{k+1}(x_{j,n})]\hat{B}_\ell(x_{j,n})\lambda_{j,n} = \delta_{k,l}, \\ 0 \leqslant k < n-1, \quad 0 \leqslant \ell \leqslant n-1, \qquad (2.8)$$

where

$$\lambda_{j,n} = \frac{1}{\alpha_{n+1}\hat{B}_n'(x_{j,n})\tilde{B}_{n-1}(x_{j,n})} > 0.$$

Taking $k = \ell = 0$ in (2.8) gives

$$\beta_1 \sum_{j=1}^{n} \frac{\lambda_{j,n}}{x_{j,n}} = 1,$$

taking $k = 0$ and $\ell = 1$ gives after some simplification

$$\sum_{j=1}^{n} \lambda_{j,n} = 1,$$

and taking $k = 0$ and $\ell = 2$ gives

$$\sum_{j=1}^{n} \lambda_{j,n} x_{j,n} = \beta_1 + \alpha_2.$$

This means that we can introduce the sequence of discrete probability measures

$$\mu_n = \sum_{j=1}^{n} \lambda_{j,n} \delta_{x_{j,n}},$$

where δ_c is the Dirac measure concentrated at the point c, and (2.8) becomes

$$\int [\tilde{B}_k(x) - \tilde{B}_{k+1}(x)]\hat{B}_\ell(x)\, d\mu_n(x) = \delta_{k,l}, \qquad 0 \leqslant k < n, \quad 0 \leqslant \ell \leqslant n. \qquad (2.9)$$

Observe that the Laurent polynomials $\{\tilde{B}_k - \tilde{B}_{k+1} : 0 \leqslant k < \ell\}$ are a basis for the linear space spanned by $\{x^{-1}, x^{-2}, \ldots, x^{-\ell}\}$, hence by linearity (2.9) implies for $\ell \leqslant n$

$$\int x^{-\ell+s} B_\ell(x)\, d\mu_n(x) = 0, \qquad s = 0, 1, \ldots, \ell - 1. \qquad (2.10)$$

Now note that the Laurent polynomials $\{[\tilde{B}_k - \tilde{B}_{k+1}]\hat{B}_\ell : 0 \leqslant k < n,\ 0 \leqslant \ell \leqslant n\}$ span the space of all Laurent polynomials of the form $f(x) = \sum_{k=-n}^{n-1} a_k x^k$, hence by linearity all the moments

$$\int x^m\, d\mu_n(x), \qquad -n \leqslant m < n$$

exist and

$$\int x^m \, d\mu_n(x) = \int x^m \, d\mu_{n+k}(x), \qquad k \geqslant 0. \tag{2.11}$$

The sequence of probability measures μ_n is tight, i.e., for every $\varepsilon > 0$ there exists a compact interval $[A, B]$ such that $\mu_n([A, B]) > 1 - \varepsilon$ for every n. Indeed, we have on one hand for $A > 0$

$$\mu_n([0, A)) = \sum_{x_{j,n} < A} \lambda_{j,n} < A \sum_{j=1}^{n} \frac{\lambda_{j,n}}{x_{j,n}} = \frac{A}{\beta_1},$$

and on the other hand for $B > 0$

$$\mu_n((B, \infty)) = \sum_{x_{j,n} > B} \lambda_{j,n} < \frac{1}{B} \sum_{j=1}^{n} \lambda_{j,n} x_{j,n} = \frac{\beta_1 + \alpha_2}{B},$$

so that

$$\mu_n([A, B]) = 1 - \mu_n([0, A)) - \mu_n((B, \infty)) > 1 - \frac{A}{\beta_1} - \frac{\beta_1 + \alpha_2}{B}.$$

For every $\varepsilon > 0$ we can therefore take $A = \varepsilon\beta_1/2$ and $B = (\beta_1 + \alpha_2)/(2\varepsilon)$ to find that $\mu_n([A, B]) > 1 - \varepsilon$ for every n. This tightness implies that this sequence of measures has a subsequence that converges to a probability measure μ (see, e.g., [2]) and all the strong moments of μ exist because of (2.11). Taking the limit in (2.8) over this subsequence gives

$$\int [\tilde{B}_k(x) - \tilde{B}_{k+1}(x)] \hat{B}_\ell(x) \, d\mu(x) = \delta_{k,\ell}, \qquad k, \ell \geqslant 0,$$

and the limit of (2.10) is precisely the orthogonality given in (1.2). This gives a proof of this orthogonality property without the use of positive T-fractions [6]. The measure μ need not be unique: this will depend on whether or not the strong Stieltjes moment problem is determinate. We will show later that (1.3) implies that μ is unique and is supported on a compact subset of $(0, \infty)$.

We can now write (2.8) as

$$\sum_{j=1}^{n} [\tilde{B}_k(x_{j,n}) - \tilde{B}_{k+1}(x_{j,n})] \hat{B}_\ell(x_{j,n}) \lambda_{j,n}$$
$$= \int [\tilde{B}_k(x) - \tilde{B}_{k+1}(x)] \hat{B}_\ell(x) \, d\mu(x), \qquad 0 \leqslant k < n, \quad 0 \leqslant \ell \leqslant n$$

and since $\{[\tilde{B}_k - \tilde{B}_{k+1}]\hat{B}_\ell : 0 \leqslant k < n-1,\ 0 \leqslant \ell \leqslant n-1\}$ span the space of all Laurent polynomials of the form $f(x) = \sum_{k=-n}^{n-1} a_k x^k$, we have by linearity

$$\sum_{j=1}^{n} \lambda_{j,n} f(x_{j,n}) = \int f(x)\, d\mu(x), \qquad f(x) = \sum_{k=-n}^{n-1} a_k x^k. \tag{2.12}$$

This is the Gauss quadrature formula for Laurent polynomials, found earlier in [12], which is of course a consequence of the Gaussian quadrature formula for the orthogonal polynomials of degree n for the varying weight $x^{-n}\, d\mu(x)$.

3. LOCATION OF THE ZEROS

The zeros of B_n are real, positive and simple. Furthermore, from the recurrence relation (1.1) it follows easily that the zeros of B_n and B_{n-1} interlace. Indeed, at a zero of B_{n-1} the polynomials B_n and B_{n-2} have opposite signs. Hence if B_{n-1} and B_{n-2} have interlacing zeros, then B_{n-2} changes sign at each zero of B_{n-1}, and therefore also B_n changes sign at each zero of B_{n-1}. Since $B_n(0) = (-1)^n \beta_n \beta_{n-1} \cdots \beta_1$ and the sign of B_n near $+\infty$ is positive, it follows therefore by induction that there is a zero between two consecutive zeros of B_{n-1} and furthermore there is a zero between 0 and the smallest zero of B_{n-1} and a zero to the right of the largest zero of B_{n-1}. This interlacing property was already observed in [6].

If we shift all the recurrence coefficients by one, then we get the associated polynomials $B_n^{(1)}$ which satisfy the recurrence relation

$$B_{n-1}^{(1)}(x) = (x - \beta_n) B_{n-2}^{(1)}(x) - \alpha_n x B_{n-3}^{(1)}(x), \tag{3.1}$$

with $B_0^{(1)} = 1$ and $B_{-1}^{(1)} = 0$. Multiply (1.1) by $B_{n-2}^{(1)}(x)$ and (3.1) by $B_{n-1}(x)$ and subtract the two obtained equations, then one finds

$$B_{n-2}^{(1)}(x) B_n(x) - B_{n-1}^{(1)}(x) B_{n-1}(x) = -x^{n-1} \alpha_n \alpha_{n-1} \cdots \alpha_2.$$

Hence, at a zero of B_{n-1} the polynomials $B_{n-2}^{(1)}$ and B_n have opposite signs. This gives $n-1$ sign changes for $B_{n-2}^{(1)}$ so that the zeros of $B_{n-2}^{(1)}$ and B_{n-1} are interlacing. Of course, this fact is not new either, but it will be of use to us later.

THEOREM 3.1. *Suppose that* $\alpha_n \leqslant M$ *and* $0 < N \leqslant \beta_n \leqslant M$ *for every* n. *Then all the zeros of* B_n *are in a compact interval* $[A, B]$ *with* $A > 0$. *Furthermore, if*

$$\lim_{n\to\infty} \alpha_n = \alpha > 0, \qquad \lim_{n\to\infty} \beta_n = \beta > 0.$$

Then for every $0 < \varepsilon < 1/3$ *there exists an integer* m, *independent of* n, *such that* B_n *has at most* m *zeros to the left of* $(1-3\varepsilon)(\sqrt{\alpha+\beta}-\sqrt{\alpha})^2$ *and at most* m *zeros to the right of* $(1+\varepsilon)(\sqrt{\alpha+\beta}+\sqrt{\alpha})^2$.

Proof. We will use the fact that the zeros of B_n are also the eigenvalues of the matrix $G^{-1}F$ given in (2.2), but we will apply a similarity transformation to this matrix. Let D_a be the diagonal matrix with diagonal $(a, a^2, a^3, \ldots, a^n)$, where $a > 0$, and consider the matrix $D_a G^{-1} F D_a^{-1}$, then this matrix has the same eigenvalues as $G^{-1}F$. An upper bound for the eigenvalues is the spectral radius $\rho(D_a G^{-1} F D_a^{-1})$. Now $\rho(A) \leqslant \|A\|_1$, where $\|\cdot\|_1$ is the matrix norm

$$\|A\|_1 = \max_{1 \leqslant j \leqslant n} \sum_{i=1}^{n} |A_{i,j}|.$$

Hence the matrix norm $\|D_a G^{-1} F D_a^{-1}\|_1$ gives an upper bound for the zeros of B_n. We have

$$\|D_a G^{-1} F D_a^{-1}\|_1 \leqslant \|D_a G^{-1} D_a^{-1}\|_1 \|D_a F D_a^{-1}\|_1.$$

One easily finds

$$D_a F D_a^{-1} = \begin{pmatrix} \beta_1 & 0 & 0 & \cdots & 0 \\ a\alpha_2 & \beta_2 & 0 & \cdots & 0 \\ 0 & a\alpha_3 & \beta_3 & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a\alpha_n & \beta_n \end{pmatrix}$$

so that $\|D_a F D_a^{-1}\|_1 \leqslant M(1+a)$. On the other hand

$$D_a G^{-1} D_a^{-1} = \begin{pmatrix} 1 & 1/a & 1/a^2 & \cdots & 1/a^{n-2} & 1/a^{n-1} \\ 0 & 1 & 1/a & \cdots & 1/a^{n-3} & 1/a^{n-2} \\ 0 & 0 & 1 & \cdots & 1/a^{n-4} & 1/a^{n-3} \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 1/a \\ 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix},$$

so that $\|D_a G^{-1} D_a^{-1}\|_1 \leqslant \frac{a}{a-1}$ whenever $a > 1$. The zeros $x_{j,n}$ of B_n, therefore, satisfy

$$x_{j,n} \leqslant M \frac{(1+a)a}{a-1},$$

for any $a > 1$. Taking $a = 2$ gives $x_{j,n} \leqslant 6M$ as an upperbound. To find a lower bound for the zeros, we consider the matrix $F^{-1}G$ for which the eigenvalues are $1/x_{j,n}$ ($j = 1, 2, \ldots, n$). We now find that $\rho(D_a F^{-1} G D_a^{-1}) \leqslant \| D_a F^{-1} D_a^{-1}\|_1 \, \|D_a G D_a^{-1}\|_1$, and one easily finds $\|D_a G D_a^{-1}\|_1 \leqslant (1+a)/a$ for any $a > 0$. The matrix F^{-1} is a lower triangular matrix with entries

$$(F^{-1})_{i,j} = (-1)^{i-j} \frac{\prod_{k=j+1}^{i} \alpha_k}{\prod_{k=j}^{i} \beta_k}, \qquad i \geqslant j,$$

so that $|(F^{-1})_{i,j}| \leqslant \frac{M^{i-j}}{N^{i-j+1}}$ and

$$\|D_a F^{-1} D_a^{-1}\|_1 \leqslant \frac{1}{N - aM},$$

for any $a < N/M$. This gives the upper bound for the reciprocals of the zeros

$$\frac{1}{x_{j,n}} \leqslant \frac{1+a}{a(N-aM)},$$

and thus the lower bound

$$x_{j,n} \geqslant \frac{a(N-aM)}{1+a},$$

for any a such that $0 < a < N/M$. Taking $2a = N/M$ gives $x_{j,n} \geqslant N^2/(4M + 2N)$. Hence the first part of the theorem holds with $A = N^2/(4M + 2N)$ and $B = 6M$.

For the second part of the theorem we consider the zeros $x_{j,n-m}^{(m)}$ of the polynomials $B_{n-m}^{(m)}$. These are the eigenvalues of the matrix $G^{-1}F_m$, where we are now dealing with matrices of order $n - m$ and

$$F_m = \begin{pmatrix} \beta_{m+1} & 0 & 0 & \cdots & 0 \\ \alpha_{m+2} & \beta_{m+2} & 0 & \cdots & 0 \\ 0 & \alpha_{m+3} & \beta_{m+3} & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \alpha_n & \beta_n \end{pmatrix}.$$

For $0<\varepsilon<1/3$ we choose m such that

$$|\alpha_k - \alpha| \leqslant \alpha\varepsilon, \qquad |\beta_k - \beta| \leqslant \beta\varepsilon, \quad k \geqslant m.$$

Then $\alpha_k \leqslant \alpha(1+\varepsilon)$, $\beta_k \leqslant \beta(1+\varepsilon)$ and $\beta_k \geqslant \beta(1-\varepsilon)$ for $k \geqslant m$. We can then obtain bounds as in the first part of this theorem, namely for $a > 1$

$$\|D_a F_m D_a^{-1}\|_1 \leqslant (\beta + a\alpha)(1+\varepsilon), \qquad \|D_a G^{-1} D_a^{-1}\|_1 \leqslant \frac{a}{a-1},$$

which gives

$$x_{j,n-m}^{(m)} \leqslant \frac{a(\beta + a\alpha)}{a-1}(1+\varepsilon).$$

The minimum of the right-hand side is obtained for $a = 1 + \sqrt{\alpha+\beta}/\sqrt{\alpha}$ and gives

$$x_{j,n-m}^{(m)} \leqslant (\sqrt{\alpha+\beta} + \sqrt{a})^2(1+\varepsilon).$$

For a lower bound we look at the eigenvalues of $F_m^{-1}G$ and find that

$$\|D_a F_m^{-1} D_a^{-1}\|_1 \leqslant \frac{1}{\beta(1-\varepsilon) - a\alpha(1+\varepsilon)},$$

whenever $a \leqslant \frac{\beta(1-\varepsilon)}{\alpha(1+\varepsilon)}$. Since $\varepsilon \leqslant 1/3$ we have that $(1-\varepsilon)/(1+\varepsilon) \geqslant \frac{1}{2}$, hence we will take $a \leqslant \frac{\beta}{2\alpha}$. Together with $\|D_a G D_a^{-1}\|_1 \leqslant (1+a)/a$ this gives

$$\frac{1}{x_{j,n-m}^{(m)}} \leqslant \frac{1+a}{a(\beta - a\alpha)}\left(1 - \varepsilon\frac{\beta + a\alpha}{\beta - a\alpha}\right)^{-1}.$$

The minimum of $(1+a)/[a(\beta - a\alpha)]$ is obtained for $a = -1 + \sqrt{\alpha+\beta}/\sqrt{\alpha}$, which is positive and less than $\beta/(2\alpha)$, and gives

$$x_{j,n-m}^{(m)} \geqslant (\sqrt{\alpha+\beta} - \sqrt{\alpha})^2\left(1 - \varepsilon\frac{\beta + a\alpha}{\beta - a\alpha}\right).$$

A simple calculation, and the arithmetic–geometric mean inequality, shows that for this value of a

$$\frac{\beta + a\alpha}{\beta - a\alpha} \leqslant 3,$$

hence

$$x_{j,n-m}^{(m)} \geqslant (\sqrt{\alpha+\beta} - \sqrt{\alpha})^2(1 - 3\varepsilon).$$

By using the interlacing property, $B^{(m-1)}_{n-m+1}$ has at most one zero to the left of the interval $[(\sqrt{\alpha+\beta}-\sqrt{\alpha})^2(1-3\varepsilon), (\sqrt{\alpha+\beta}+\sqrt{\alpha})^2(1+\varepsilon)]$, and one zero to the right of that interval. Continuing this way, it follows that B_n has at most m zeros to the left of the interval and m zeros to the right, which is what we wanted to prove. ∎

One consequence of this theorem is that the Laurent orthogonality measure μ will have a support in $[A, B]$, with $A > 0$, whenever $\alpha_n \leqslant M$ and $0 < N \leqslant \beta_n \leqslant M$ for all n, since we obtained the measure μ as a weak limit of measures μ_n supported on the zeros. All these measures are supported on $[A, B]$, hence also any weak limit. The Laurent orthogonality measure is unique in this case.

From the interlacing of the zeros of B_n and B_{n-1}, it follows that the sequence of the kth zeros $x_{k,n}$ is decreasing with n and the sequence of the kth largest zeros $x_{n-k+1,n}$ is increasing with n. Hence

$$\lim_{n\to\infty} x_{k,n} = X_k, \qquad \lim_{n\to\infty} x_{n-k+1,n} = Y_k$$

exist, and X_k is increasing and Y_k is decreasing. Theorem 3.1 now implies that for each $0 < \varepsilon < 1/3$ there exists an integer m such that $X_k > (1-3\varepsilon)(\sqrt{\alpha+\beta}-\sqrt{\alpha})^2$ and $Y_k < (1+\varepsilon)(\sqrt{\alpha+\beta}+\sqrt{\alpha})^2$ for $k > m$. In particular, this means that

$$\lim_{k\to\infty} X_k \geqslant (\sqrt{\alpha+\beta}-\sqrt{\alpha})^2, \qquad \lim_{k\to\infty} Y_k \leqslant (\sqrt{\alpha+\beta}+\sqrt{\alpha})^2. \tag{3.2}$$

4. RATIO ASYMPTOTICS

In order to get more detailed information about the set where the zeros of B_n $(n = 0, 1, 2, \ldots)$ are dense, we will investigate the ratio asymptotic behaviour of the Laurent polynomials given by recurrence (1.1).

THEOREM 4.1. *Suppose that*

$$\lim_{n\to\infty} \alpha_n = \alpha > 0, \qquad \lim_{n\to\infty} \beta_n = \beta > 0.$$

Then

$$\lim_{n\to\infty} \frac{B_{n-1}(x)}{B_n(x)} = \frac{2}{x-\beta+\sqrt{(x-\beta)^2-4x\alpha}} \tag{4.1}$$

holds uniformly on compact subsets of $\mathbb{C}\backslash([a,b] \cup \{X_k, Y_k : k \in \mathbb{N}\})$, *where* $a = (\sqrt{\alpha+\beta} - \sqrt{\alpha})^2$ *and* $b = (\sqrt{\alpha+\beta} + \sqrt{\alpha})^2$ *and the square root is such that* $\sqrt{(x-\beta)^2 - 4x\alpha}$ *is an analytic function of* x *in* $\mathbb{C}\backslash[a,b]$, *with* $\sqrt{(x-\beta)^2 - 4x\alpha} > 0$ *if* $x > b$ *and* $\sqrt{(x-\beta)^2 - 4x\alpha} < 0$ *for* $x < a$.

Proof. This is a consequence of Poincaré's theorem (see, e.g., [7, 8]) which states that for a sequence y_n $(n = 0, 1, 2, \ldots)$ satisfying a linear recurrence relation of the form $a_n y_n + b_n y_{n-1} + c_n y_{n-2} = 0$ with converging recurrence coefficients $a_n \to a \neq 0$, $b_n \to b$, $c_n \to c \neq 0$, the ratio y_{n-1}/y_n converges to a solution z of the characteristic equation $a + bz + cz^2$ provided both roots z_1 and z_2 have a different modulus ($|z_1| \neq |z_2|$). In the case under consideration, the characteristic equation is

$$x\alpha z^2 - (x-\beta)z + 1 = 0,$$

and the roots are

$$z_1 = \frac{x - \beta - \sqrt{(x-\beta)^2 - 4x\alpha}}{2x\alpha} = \frac{2}{x - \beta + \sqrt{(x-\beta)^2 - 4x\alpha}},$$

$$z_2 = \frac{x - \beta + \sqrt{(x-\beta)^2 - 4x\alpha}}{2x\alpha} = \frac{2}{x - \beta - \sqrt{(x-\beta)^2 - 4x\alpha}}.$$

Observe that for large x the ratio $B_{n-1}(x)/B_n(x)$ behaves like $1/x$, hence for x large we have to take the root z_1 which also behaves like $1/x$. The zeros of B_{n-1} and B_n interlace, hence we have the partial fraction decomposition

$$\frac{B_{n-1}(x)}{B_n(x)} = \sum_{j=1}^{n} \frac{a_{j,n}}{x - x_{j,n}},$$

with positive residues $a_{j,n}$ for which $\sum_{j=1}^{n} a_{j,n} = 1$. If x is in a compact subset K of $\mathbb{C}\backslash([a,b] \cup \{X_k, Y_k : k \in \mathbb{N}\})$, then there exist a $\delta > 0$ such that $|x - x_{j,n}| > \delta$ for $0 \leqslant j \leqslant n$ and n large enough, since all accumulation points of zeros of B_n are in $[a,b] \cup \{X_k, Y_k : k \in \mathbb{N}\}$. Hence

$$\left|\frac{B_{n-1}(x)}{B_n(x)}\right| \leqslant \sum_{j=1}^{n} \frac{a_{j,n}}{|x - x_{j,n}|} < \frac{1}{\delta}, \qquad x \in K,$$

so that the ratios B_{n-1}/B_n are a normal family on K. We already know that this ratio converges to z_1 for large x, hence the Stieltjes–Vitali theorem implies that it converges to z_1 uniformly on every compact subset of $\mathbb{C}\backslash([a, b] \cup \{X_k, Y_k : k \in \mathbb{N}\})$ and the limit function must be an analytic function on $\mathbb{C}\backslash([a, b] \cup \{X_k, Y_k : k \in \mathbb{N}\})$. ∎

Observe that the roots z_1 and z_2 have equal modulus when $(x-\beta)^2 - 4x\alpha \leqslant 0$, and this is precisely the interval $[a, b]$. We are now ready to prove Blumenthal's theorem for Laurent orthogonal polynomials.

THEOREM 4.2. *Suppose that*

$$\lim_{n\to\infty} \alpha_n = \alpha > 0, \qquad \lim_{n\to\infty} \beta_n = \beta > 0.$$

Then the zeros of the Laurent orthogonal polynomials B_n are dense on $[(\sqrt{\alpha+\beta} - \sqrt{\alpha})^2, (\sqrt{\alpha+\beta} + \sqrt{\alpha})^2] \cup \{X_k, Y_k : k \in \mathbb{N}\}$. The set $\{X_k, Y_k : k \in \mathbb{N}\}$ is either finite or has accumulation points at $a = (\sqrt{\alpha+\beta} - \sqrt{\alpha})^2$ and/or $b = (\sqrt{\alpha+\beta} + \sqrt{\alpha})^2$.

Proof. The limit function in (4.1) has singularities on the interval $[a, b]$, and the ratio B_{n-1}/B_n has its singularities at the zeros of B_n. Suppose that $x_0 \in [a, b]$ is not an accumulation point of zeros, then there exists $\varepsilon > 0$ and an integer n_0 such that $(x_0 - \varepsilon, x_0 + \varepsilon)$ contains no zeros of B_n for $n > n_0$. But then in the ball $D(x_0, \varepsilon/2) = \{|z - x_0| < \varepsilon/2\}$ we have $|z - x_{j,n}| > \varepsilon/2$, so that $|B_{n-1}(z)/B_n(z)| < 2/\varepsilon$ on $D(x_0, \varepsilon/2)$, as in the proof of the previous theorem. The ratio $B_{n-1}(z)/B_n(z)$ is, therefore, a normal family on $D(x_0, \varepsilon/2)$, so that there exists a subsequence that converges to an analytic function. But this is in contradiction with (4.1), hence every point on $[a, b]$ is an accumulation point of zeros. Obviously each X_k and Y_k is an accumulation point, since they are defined as limits of extreme zeros. This covers all the possibilities if we take (3.2) into account. ∎

5. THE SUPPORT OF THE MEASURE

First we will give the asymptotic distribution of the zeros of the polynomials B_n, which follows as a consequence of Theorem 4.1.

THEOREM 5.1. *Suppose that*

$$\lim_{n\to\infty} \alpha_n = \alpha > 0, \qquad \lim_{n\to\infty} \beta_n = \beta > 0.$$

Then

$$\lim_{n\to\infty} \frac{1}{n} \frac{B_n'(x)}{B_n(x)} = \frac{x + \beta + \sqrt{(x-\beta)^2 - 4\alpha x}}{2x\sqrt{(x-\beta)^2 - 4\alpha x}} \tag{5.1}$$

holds uniformly on compact subsets of $\mathbb{C}\backslash([a,b] \cup \{X_k, Y_k : k \in \mathbb{N}\})$, *where* $a = (\sqrt{\alpha+\beta} - \sqrt{\alpha})^2$ *and* $b = (\sqrt{\alpha+\beta} + \sqrt{\alpha})^2$.

Proof. We can take derivatives in Theorem 4.1 to find

$$\lim_{n\to\infty} \frac{B_{n-1}(x)}{B_n(x)}\left(\frac{B'_{n-1}(x)}{B_{n-1}(x)} - \frac{B'_n(x)}{B_n(x)}\right) = G'(x),$$

where $G'(x)$ is the derivative of the right-hand side of (4.1). This means that

$$\lim_{n\to\infty}\left(\frac{B'_n(x)}{B_n(x)} - \frac{B'_{n-1}(x)}{B_{n-1}(x)}\right) = -\frac{G'(x)}{G(x)}.$$

A simple calculation shows that $-G'(x)/G(x)$ is equal to the expression on the right-hand side of (5.1). Cesàro's lemma then gives the desired result. ∎

COROLLARY 5.1. *Suppose that*

$$\lim_{n\to\infty} \alpha_n = \alpha > 0, \qquad \lim_{n\to\infty} \beta_n = \beta > 0.$$

Then the asymptotic distribution of the zeros $x_{1,n} < x_{2,n} < \cdots < x_{n,n}$ *of* B_n *is given by the density*

$$v(t) = \frac{1}{2\pi}\frac{t+\beta}{t\sqrt{4\alpha t - (t-\beta)^2}}, \qquad t \in [a,b], \tag{5.2}$$

where $a = (\sqrt{\alpha+\beta} - \sqrt{\alpha})^2$ *and* $b = (\sqrt{\alpha+\beta} + \sqrt{\alpha})^2$. *This means that for every continuous function* f *on* $\mathbb{R}^+$ *we have*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f(x_{k,n}) = \int_a^b f(t)v(t)\,dt.$$

Proof. Let v_n be the nth order zero counting measure

$$v_n = \frac{1}{n}\sum_{k=1}^{n} \delta_{x_{k,n}},$$

where $\delta_{x_{k,n}}$ is the Dirac measure with unit mass at the zero $x_{k,n}$. Then Theorem 5.1 implies that

$$\lim_{n\to\infty} \int \frac{dv_n(t)}{x-t} = \frac{x+\beta+\sqrt{(x-\beta)^2 - 4\alpha x}}{2x\sqrt{(x-\beta)^2 - 4\alpha x}}, \tag{5.3}$$

uniformly for $x \in \mathbb{C}\setminus([a,b] \cup \{X_k, Y_k : k \in \mathbb{N}\})$. If we use the integral

$$\frac{1}{\pi}\int_a^b \frac{1}{\sqrt{4\alpha t - (t-\beta)^2}}\frac{dt}{x-t} = \frac{1}{\sqrt{(x-\beta)^2 - 4\alpha x}}, \qquad x \in \mathbb{C}\setminus[a,b],$$

where the square root is such that $\sqrt{(x-\beta)^2 - 4\alpha x} > 0$ for $x > b$ and $\sqrt{(x-\beta)^2 - 4\alpha x} < 0$ for $x < a$, then one easily verifies that the right-hand side in (5.3) is given by

$$\frac{x + \beta + \sqrt{(x-\beta)^2 - 4\alpha x}}{2x\sqrt{(x-\beta)^2 - 4\alpha x}} = \int_a^b \frac{v(t)}{x-t}\,dt.$$

This convergence of the Stieltjes transforms implies the weak convergence of the zero counting measures v_n using the Grommer–Hamburger convergence theorem [15, p. 175]. ∎

The limiting measure with density given by (5.2) is the convex combination $v(t) = (v_0(t) + v_1(t))/2$ of the arcsine weight on $[a,b]$

$$v_0(t) = \frac{1}{\pi}\frac{1}{\sqrt{4\alpha t - (t-\beta)^2}}, \qquad t \in [a,b],$$

and the weight

$$v_1(t) = \frac{\beta}{\pi}\frac{1}{t\sqrt{4\alpha t - (t-\beta)^2}}, \qquad t \in [a,b].$$

The latter is in fact the balayage of the Dirac measure δ_0 on $[a,b]$ (see, e.g., [11, Eq. (4.47), p. 122]). The asymptotic zero distribution given by Corollary 5.1 is, therefore, compatible with the zero distribution that would follow using logarithmic potential theory and properties of the orthogonality measure (such as regularity of the measure μ on the interval $[a,b]$). Indeed, the polynomials B_n are orthogonal polynomials of degree n for the varying measure $d\mu_n(x) = x^{-n}\,d\mu(x)$. This is a problem of weighted polynomials, with external field $Q(x) = -\frac{1}{2n}\log x^{-n} = \frac{1}{2}\log x$. This external field is equal to $Q(x) = -\frac{1}{2}U(x;\delta_0)$, where $U(v,x)$ is the logarithmic potential of the measure v. The asymptotic zero distribution is then given by the equilibrium measure in $[a,b]$ with external field Q, and according to [11, Example 4.8, p. 118] this is the convex combination $v(t) = (v_0(t) + v_1(t))/2$.

Next we will show that the set $[(\sqrt{\alpha+\beta}-\sqrt{\alpha})^2, (\sqrt{\alpha+\beta}+\sqrt{\alpha})^2] \cup \{X_k, Y_k : k \in \mathbb{N}\}$, on which the zeros are dense, is very closely related to the support of the measure μ. Indeed, we have

THEOREM 5.2. *Suppose that*

$$\lim_{n\to\infty} \alpha_n = \alpha > 0, \qquad \lim_{n\to\infty} \beta_n = \beta > 0.$$

Then the support of the orthogonality measure μ *in* (1.2) *is the interval* $[(\sqrt{\alpha+\beta}-\sqrt{\alpha})^2, (\sqrt{\alpha+\beta}+\sqrt{\alpha})^2]$ *together with the set* $\{X_k, Y_k : k \in \mathbb{N}\}$, *which is either finite or has accumulation points at* $(\sqrt{\alpha+\beta}-\sqrt{\alpha})^2$ (*for* X_k) *and/or* $(\sqrt{\alpha+\beta}+\sqrt{\alpha})^2$ (*for* Y_k).

This result can be proved in a similar way as the corresponding result for the usual orthogonal polynomials (see, e.g., [8, 9]). We will prove this result using a lemma which extends well-known properties of zeros of orthogonal polynomials [5, Theorem 4.3; 14, Theorem 6.1.1]) to Laurent orthogonality.

LEMMA 5.1. (1) *Suppose* (c, d) *is an interval where* μ *has no mass, i.e.,* $\mu(c, d) = 0$. *Then there is at most one zero of each polynomial* B_n *in* (c, d).

(2) *Suppose that the support of* μ *is compact and that* (c, d) *is an interval such that* $\mu(c, d) > 0$. *Then for n sufficiently large, the polynomial* B_n *will have at least one zero in* (c, d).

Proof. (1) Suppose B_n has two zeros x_1 and x_2 in (c, d), then we can write $B_n(x) = (x - x_1)(x - x_2)C_{n-2}(x)$, where C_{n-2} is a polynomial of degree $n - 2$. Consider the integral

$$I_n = \int x^{-n} B_n(x) C_{n-2}(x)\, d\mu(x),$$

then by the Laurent orthogonality (1.2) we have that $I_n = 0$. On the other hand, we have $(x - x_1)(x - x_2) > 0$ whenever $x \in \mathbb{R} \setminus (c, d)$, hence

$$I_n = \int_{\mathbb{R}\setminus(c,d)} x^{-n}(x - x_1)(x - x_2) C_{n-2}^2(x)\, d\mu(x) > 0,$$

and this contradiction implies that B_n cannot have two zeros in (c, d).

(2) Suppose that for every $n > n_0$ the polynomial B_n has no zeros in (c, d). Consider the function

$$f(x) = \begin{cases} 0 & \text{for } x \leqslant c \quad \text{and} \quad x \geqslant d, \\ (x - c)(d - x) & \text{for } c < x < d, \end{cases}$$

then f is a continuous and non-negative function. For every $\varepsilon > 0$ we can find a polynomial p_m such that $|f(x) - p_m(x)| < \varepsilon$ on an compact interval containing the support of μ (Weierstrass theorem). Hence

$$\int f(x)\, d\mu(x) \leqslant \int |f(x) - p_m(x)|\, d\mu(x) + \int p_m(x)\, d\mu(x)$$
$$\leqslant \varepsilon\, \mu(\mathbb{R}) + \sum_{j=1}^{n} \lambda_{j,n} p_m(x_{j,n}),$$

provided $n > m$, where we used the Gauss quadrature (2.12). The zeros $x_{j,n}$ are all outside (c, d), and in $\mathrm{supp}(\mu)\backslash(c, d)$ we have $|p_m(x)| < \varepsilon$. We thus find

$$\int f(x)\, d\mu(x) \leqslant \varepsilon\mu(\mathbb{R}) + \varepsilon \sum_{j=1}^{n} \lambda_{j,n} = 2\varepsilon\mu(\mathbb{R}).$$

The right-hand side can be made as small as possible, but the left-hand side is a positive quantity. This contradiction implies that for sufficiently large n the polynomial B_n has a zero in (c, d). ∎

Proof of Theorem 5.2. First we prove that $[a, b] \subset \mathrm{supp}(\mu)$. Let $x \in (a, b)$ and choose ε such that $(x - \varepsilon, x + \varepsilon) \subset [a, b]$. Then the result on the zero distribution (Corollary 5.1) implies that the number of zeros of B_n in $(x - \varepsilon, x + \varepsilon)$ is equal to

$$n \int_{x-\varepsilon}^{x+\varepsilon} v(t)\, dt + o(n),$$

and hence, since $v(t) > 0$ on (a, b), for n large $(x - \varepsilon, x + \varepsilon)$ contains more than one zero of B_n. Therefore, Lemma 5.1 implies that $\mu(x - \varepsilon, x + \varepsilon) > 0$ for every $\varepsilon > 0$, and hence $x \in \mathrm{supp}(\mu)$. A similar argument works for $x = a$ and $x = b$ by considering $[a, a + \varepsilon)$ and $(b - \varepsilon, b]$, respectively. Hence $[a, b] \subset \mathrm{supp}(\mu)$.

Next we prove that each X_k and Y_k belongs to the support of μ. We limit the discussion to X_k because the reasoning for Y_k is similar. We will prove by induction on k that each X_k belongs to the support of μ. The first X_1 is the limit of $x_{1,n}$. Suppose $X_1 \notin \mathrm{supp}(\mu)$. Then there exists $\varepsilon > 0$ such that $\mu(X_1 - \varepsilon, X_1 + \varepsilon) = 0$. But the zeros of orthogonal polynomials always are in the convex hull of the support of μ, hence there should not be any zeros of B_n in $(-\infty, X_1 + \varepsilon)$, but this is in contradiction with the fact that $x_{1,n} \to X_1$. Therefore $X_1 \in \mathrm{supp}(\mu)$. Suppose that $X_1, \ldots, X_{k-1} \in \mathrm{supp}(\mu)$. If $X_k = X_{k-1}$, then there is nothing to prove. Hence assume that $X_k > X_{k-1}$ and that $X_k \notin \mathrm{supp}(\mu)$. Then there exists $\varepsilon > 0$ such that $\mu(X_k - \varepsilon, X_k + \varepsilon) = 0$. The interval (X_{k-1}, X_k) contains at most one zero of B_n, namely $x_{k-1,n}$, and this zero converges to X_{k-1}. Hence $(X_{k-1}, X_k) \cap \mathrm{supp}(\mu)$ is empty, since each point in the support of μ is an accumulation point of zeros. We see that $\mu(X_{k-1}, X_k + \varepsilon) = 0$ and hence Lemma 5.1 implies that there can be at most one zero of B_n in $(X_{k-1}, X_k + \varepsilon)$. But we already know that, for large n, $x_{k-1,n}$ is in this interval and since $x_{k,n} \to X_k$ we conclude that also $x_{k,n}$ belongs to $(X_{k-1}, X_k + \varepsilon)$ for large n. This contradiction implies that $X_k \in \operatorname{supp}(\mu)$. ∎

6. PERTURBATION

Given $\alpha > 0$ and $\beta > 0$ the Laurent–Chebyshev polynomials $B_n^{(T)}$ are those with the recurrence relation

$$B_2^{(T)}(x) = (x-\beta)B_1^{(T)}(x) - 2\alpha x B_0^{(T)}(x),$$

$$B_n^{(T)}(x) = (x-\beta)B_{n-1}^{(T)}(x) - \alpha x B_{n-2}^{(T)}(x), \qquad n \geqslant 3,$$

with $B_1^{(T)}(x) = x - \beta$, $B_0^{(T)} = 1$ and $B_{-1}^{(T)} = 0$. It was shown in [12] that these polynomials satisfy the orthogonality property

$$\frac{1}{\pi}\int_a^b x^{-n+k} B_n^{(T)}(x) \frac{dx}{\sqrt{(b-t)(t-a)}} = 2\alpha^n \delta_{k,n}, \qquad k = 0, 1, \ldots, n-1, \quad n \geqslant 1,$$

where $b = \beta^2/a = (\sqrt{\alpha+\beta} + \sqrt{\alpha})^2$.

A system of Laurent orthogonal polynomials $B_n(x)$ for which the recurrence coefficients in (1.1) satisfy

$$\lim_{n\to\infty} \alpha_n = \alpha > 0 \qquad \text{and} \qquad \lim_{n\to\infty} \beta_n = \beta > 0, \tag{6.1}$$

will be said to be a perturbation of the system of Laurent–Chebyshev polynomials. For such a system the following result holds.

THEOREM 6.1. *Suppose that* (6.1) *holds. Let* $[A, B]$ *be the smallest interval that contains the support of the positive measure* μ *associated with the orthogonality* (1.2). *Then for every* $f \in C[A, B]$ *and* $k \geqslant 0$ *one has*

$$\lim_{n\to\infty} \int f(x)\breve{B}_n(x)\breve{B}_{n+k}(x)\, d\mu(x) \to \frac{1}{\pi}\int_a^b f(u)\breve{B}_k^{(T)}(u)\frac{du}{\sqrt{(b-u)(u-a)}}.$$

Here $\breve{B}_0^{(T)} = 1$, $\breve{B}_0 = 1$,

$$\breve{B}_n^{(T)}(x) = \frac{B_n^{(T)}(x)}{2(\sqrt{\alpha x})^n} \quad \textit{and} \quad \breve{B}_n(x) = \frac{B_n(x)}{\sqrt{\alpha_2\alpha_3\cdots\alpha_{n+1}}(\sqrt{x})^n}, \qquad n \geqslant 1.$$

Proof. Consider the integrals

$$I_0^{(T)}(m,k) = \frac{1}{\pi}\int_a^b u^m \breve{B}_{2k}^{(T)}(u) \frac{1}{\sqrt{(b-u)(u-a)}}\,du,$$

$$I_{1/2}^{(T)}(m,k) = \frac{1}{\pi}\int_a^b u^{m+1/2} \breve{B}_{2k+1}^{(T)}(u) \frac{du}{\sqrt{(b-u)(u-a)}},$$

$$I_0(m,k,n) = \int x^m \breve{B}_n(x)\breve{B}_{n+2k}(x)\,d\mu(x),$$

$$I_{1/2}(m,k,n) = \int x^{m+1/2} \breve{B}_n(x)\breve{B}_{n+2k+1}(x)\,d\mu(x),$$

defined for $n \geqslant 0$, $m \geqslant 0$ and for all possible values k. Since $[A,B] \subset (0,\infty)$, the spans of $\{1, x, x^2, x^3, \ldots\}$ and $\{x^{1/2}, x^{3/2}, x^{5/2}, x^{7/2}, \ldots\}$ are both dense in $C[A,B]$. Hence we have established the theorem if we can show

$$I_0(m,k) = \lim_{n\to\infty} I_0(m,k,n) = I_0^{(T)}(m,k),$$

$$I_{1/2}(m,k) = \lim_{n\to\infty} I_{1/2}(m,k,n) = I_{1/2}^{(T)}(m,k), \tag{6.2}$$

for all $m \geqslant 0$ and $k \geqslant 0$. The limits, as $n \to \infty$, exist because of (6.1) and the orthogonality relations for the polynomials B_n.

First one verifies from the orthogonality properties of $B_n^{(T)}$ and B_n,

$$I_0^{(T)}(0,k) = \begin{cases} 1, & k = 0, \\ 0, & k \geqslant 1, \end{cases} \qquad I_{1/2}^{(T)}(0,k) = \begin{cases} \sqrt{\alpha}, & k = 0, \\ 0, & k \geqslant 1 \end{cases}$$

and for $n \geqslant 0$,

$$I_0(0,k,n) = \begin{cases} 1, & k = 0, \\ 0, & k \geqslant 1, \end{cases} \qquad I_{1/2}(0,k,n) = \begin{cases} \sqrt{\alpha_{n+2}}, & k = 0, \\ 0, & k \geqslant 1. \end{cases}$$

Hence (6.2) clearly holds for $m = 0$. From the recurrence relation for $B_n^{(T)}$,

$$I_0^{(T)}(m,0) = 2\sqrt{\alpha} I_{1/2}^{(T)}(m-1,0) + \beta I_0^{(T)}(m-1,0),$$

$$\begin{aligned} I_0^{(T)}(m,k) = {} & \sqrt{\alpha} I_{1/2}^{(T)}(m-1,k) \\ & + \beta I_0^{(T)}(m-1,k) + \sqrt{\alpha} I_{1/2}^{(T)}(m-1,k-1), \qquad k \geqslant 1, \end{aligned}$$

$$I_{1/2}^{(T)}(m,k) = \sqrt{\alpha} I_0^{(T)}(m,k+1) + \beta I_{1/2}^{(T)}(m-1,k) + \sqrt{\alpha} I_0^{(T)}(m,k), \qquad k \geqslant 0.$$

Note that these relations applied in the given order for $m = 1, 2, 3, \ldots$ generate all the integrals $I_0^{(T)}(m,k)$ and $I_{1/2}^{(T)}(m,k)$.

Now from the recurrence relation for B_n we obtain, for n large enough,

$$I_0(m,k,n) = \sqrt{\alpha_{n+2}} I_{1/2}(m-1,k-1,n+1) + \beta_{n+1} I_0(m-1,k,n) + \sqrt{\alpha_{n+1}} I_{1/2}(m-1,k,n-1), \qquad k \geqslant 0,$$

and

$$I_{1/2}(m,k,n) = \sqrt{\alpha_{n+2}} I_0(m,k,n+1) + \beta_{n+1} I_{1/2}(m-1,k,n) + \sqrt{\alpha_{n+1}} I_0(m,k+1,n-1), \qquad k \geqslant 0.$$

Letting n tends to ∞ and observing that $I_{1/2}(m,-1) = I_{1/2}(m,0)$, we obtain a set of recurrence relations for $I_{1/2}(m,k)$ and $I_0(m,k)$ identical to those of (6.3). Since the initial conditions are the same, this establishes the validity of (6.2). ∎

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